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# Magnetohydrodynamic pipe flow in a duct with sector cross section

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**Abstract.** In this paper we use the Green function method to solve the problem of steady one-dimensional flow of an incompressible, viscous, electrically conducting fluid through a pipe with sector cross section, in the presence of an applied transverse uniform magnetic field. We obtain an analytic solution in this case which other methods have not given before.

## 1. Introduction

Magnetohydrodynamic (MHD) pipe flow problems have been solved exactly only in a few cases thus far, for example, Hartmann (1937), Shercliff (1953) and Gold (1962). Although the Green function is widely used in solving electrostatic problems, and recently in dealing with condensed matter problems, there appears to have been almost no attempt to solve MHD pipe flow problems by this powerful method. In § 2 we use this method to obtain the same analytic solution as that given by other methods in the case of a circular cross section. In § 3 we use it to obtain an analytic solution in the case of a sector cross section, which other methods have not given before.

## 2. Case of circular cross section

This problem has already been solved by Gold (1962) using Fourier analysis, but now we shall use the Green function method to solve the same problem. The two basic equations for any general MHD duct flow (incompressible, steady) with circular type section (circle, circular ring, sector) and uniform applied transverse field are

$$\nabla^2 f - \alpha^2 f = 0 \tag{1}$$

$$\nabla^2 g - \alpha^2 g = 0 \tag{2}$$

where $\alpha = \frac{1}{2}M$, $M = \mu H_0 a(\sigma/\eta)^{1/2}$ is the Hartmann number and

$$f(\rho, \theta) = \exp(\alpha\rho\cos\theta)\left(v + \frac{2\alpha}{R_m}H - \frac{K\rho\cos\theta}{2\alpha}\right) \tag{3}$$

$$g(\rho, \theta) = \exp(-\alpha\rho\cos\theta)\left(v - \frac{2\alpha}{R_m}H + \frac{K\rho\cos\theta}{2\alpha}\right) \tag{4}$$

  

where $R_m = 4\pi\sigma\mu v_0 a$ is the magnetic Reynolds number, $K = K_1 a^2/v_0\eta$, $K_1 = \partial p/\partial z =$ constant and the non-dimensional variables $v = v_z/v_0$, $H = H_z/H_0$, $\rho = r/a$, where $v_0$ is some characteristic velocity, $a$ is the radius of the pipe, $H_0$ is the uniform applied transverse field, $\rho$ the fluid density, $p$ the pressure, $\eta$ the viscosity, $\mu$ the permeability, $\boldsymbol{v}$ the velocity, and $\sigma$ the electrical conductivity.

Now, we shall find the Green function for the following equation:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}G(\boldsymbol{x},\boldsymbol{x}')\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}G(\boldsymbol{x},\boldsymbol{x}')-\alpha^2 G(\boldsymbol{x},\boldsymbol{x}')$$
$$=-\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\theta-\theta'). \tag{5}$$

The $\theta$ delta function can be written in terms of orthonormal functions:

$$\delta(\theta-\theta')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[\mathrm{i}m(\theta-\theta')]. \tag{6}$$

We expand the Green function in a similar fashion:

$$G(\boldsymbol{x},\boldsymbol{x}')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[\mathrm{i}m(\theta-\theta')]g_m(\rho,\rho'). \tag{7}$$

Substitution of (6) and (7) into (5) leads to an equation for the radial Green function $g_m(\rho,\rho')$:

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right)-\left(\alpha^2+\frac{m^2}{\rho^2}\right)g_m=-\frac{4\pi}{\rho}\delta(\rho-\rho'). \tag{8}$$

For $\rho\neq\rho'$ this is just the differential equation for the modified Bessel functions, $I_m(\alpha\rho)$ and $K_m(\alpha\rho)$. Thus its solution can be written as

$$g_m(\rho,\rho')=\begin{cases}A_m I_m(\alpha\rho) & \text{for } \rho<\rho'\\ K_m(\alpha\rho)+B_m I_m(\alpha\rho) & \text{for } \rho>\rho'\end{cases}$$

where $B_m=-K_m(\alpha)/I_m(\alpha)$. By the symmetry of $g_m(\rho,\rho')$ in $\rho$ and $\rho'$, $g_m(\rho,\rho')$ becomes

$$g_m(\rho,\rho')=A_m I_m(\alpha\rho_<)[K_m(\alpha\rho_>)+B_m I_m(\alpha\rho_>)] \tag{9}$$

where $\rho_<(\rho_>)$ is the smaller (larger) of $\rho$ and $\rho'$. To determine the constant $A_m$ we must consider the effect of the delta function in (8). If we multiply both sides by $\rho$ and integrate over the interval from $\rho=\rho'_-=\rho'-\varepsilon$ to $\rho=\rho'_+=\rho'+\varepsilon$, where $\varepsilon$ is a very small positive number, we obtain

$$\left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_+-\left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_-=-\frac{4\pi}{\rho'}. \tag{10}$$

Thus there is a discontinuity in slope at $\rho=\rho'$, where $|_\pm$ means evaluated at $\rho=\rho'_\pm=\rho'\pm\varepsilon$. By direct calculation of $\mathrm{d}g_m/\mathrm{d}\rho|_+$ and $\mathrm{d}g_m/\mathrm{d}\rho|_-$ from (9), we readily obtain

$$\left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_+-\left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_-=-\frac{A_m}{\rho'} \tag{11}$$

so that

$$A_m=4\pi. \tag{12}$$

The required Green function is therefore

$$G(\boldsymbol{x}, \boldsymbol{x}') = 2 \sum_{m=-\infty}^{\infty} \exp[im(\theta-\theta')] I_m(\alpha\rho_<)[K_m(\alpha\rho_>) + B_m I_m(\alpha\rho_>)]. \tag{13}$$

On the boundary, $\rho < \rho'$, $\rho' = 1$,

$$G(x, x') = 2 \sum_{m=-\infty}^{\infty} \exp[im(\theta-\theta')] I_m(\alpha\rho)[K_m(\alpha\rho') + B_m I_m(\alpha\rho')]. \tag{14}$$

Because of

$$\frac{\mathrm{d}}{\mathrm{d}\rho'}[K_m(\alpha\rho') + B_m I_m(\alpha\rho')]\Big|_{\rho'=1} = \alpha[K'_m(\alpha) + B_m I'_m(\alpha)]$$

$$= \frac{\alpha}{I_m(\alpha)}[I_m(\alpha)K'_m(\alpha) - K_m(\alpha)I'_m(\alpha)] = -\frac{1}{I_m(\alpha)} \tag{15}$$

therefore

$$\frac{\partial G}{\partial n'} = \frac{1}{a}\frac{\partial G}{\partial \rho'}\Big|_{\rho'=1} = -\frac{2}{a}\sum_{m=-\infty}^{\infty} \exp[im(\theta-\theta')]\frac{I_m(\alpha\rho)}{I_m(\alpha)}. \tag{16}$$

The general solution to (1) with specified values of $f(\boldsymbol{x}')$ on the boundary surface is

$$f(\boldsymbol{x}) = -\frac{1}{4\pi}\oint_s f(\boldsymbol{x}')\frac{\partial G}{\partial n'}\,\mathrm{d}a'. \tag{17}$$

On the boundary surface, since there is no fluid slip at the wall $v = 0$. Furthermore, the assumption of non-conducting walls implies that $H = 0$ and therefore, from (3) and (4), we obtain the following non-homogeneous boundary conditions on $f$ and $g$:

$$f(1, \theta) = -(K/2\alpha)\cos\theta\exp(\alpha\cos\theta) \tag{18}$$

$$g(1, \theta) = (K/2\alpha)\cos\theta\exp(-\alpha\cos\theta). \tag{19}$$

Substituting (16) and (18) in (17), we obtain

$$f(\rho, \theta) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta'(-K/2\alpha)\cos\theta'\exp(\alpha\cos\theta')$$

$$\times\left(-2\sum_{m=-\infty}^{\infty}\exp[im(\theta-\theta')]\frac{I_m(\alpha\rho)}{I_m(\alpha)}\right)$$

$$= \frac{-K}{2\alpha}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\theta'\exp(\alpha\cos\theta')\,\mathrm{d}\theta'\frac{I_0(\alpha\rho)}{I_0(\alpha)}\right.$$

$$+\frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi} 2(\cos m\theta\cos m\theta' + \sin m\theta\sin m\theta')$$

$$\left.\times\cos\theta'\exp(\alpha\cos\theta')\,\mathrm{d}\theta'\frac{I_m(\alpha\rho)}{I_m(\alpha)}\right)$$

$$= \frac{-K}{2\alpha}\left(\frac{I'_0(\alpha)}{I_0(\alpha)}I_0(\alpha\rho) + 2\sum_{m=1}^{\infty}\cos m\theta\frac{I'_m(\alpha)}{I_m(\alpha)}I_m(\alpha\rho)\right). \tag{20}$$

In the last step of (20) we have used the following relation for the modified Bessel function $I_m$:

$$I'_m(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(z\cos\theta)\cos m\theta\cos\theta\,\mathrm{d}\theta \tag{21}$$

which is obtained from differentiation of the following integral expression of $I_m$:

$$I_m(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(z\cos\theta)\cos m\theta\,\mathrm{d}\theta. \tag{22}$$

Similarly, by the above-mentioned method we can readily obtain the $g(\rho,\theta)$ function:

$$g(\rho,\theta)=\frac{-K}{2\alpha}\left(\frac{I'_0(\alpha)}{I_0(\alpha)}I_0(\alpha\rho)+\sum_{m=1}^{\infty}(-1)^m 2\cos m\theta\frac{I'_m(\alpha)}{I_m(\alpha)}I_m(\alpha\rho)\right). \tag{23}$$

(20) and (23) give the same functions $v(\rho,\theta)$ and $H(\rho,\theta)$ as that given by Gold's paper.

### 3. The case of a sector cross section

The cross section of the duct is shown in figure 1. The basic equations (1) and (2) remain the same, but the boundary conditions (18) and (19) are now replaced by the following conditions:

$$f(1,\theta)=\frac{-K}{2\alpha}\cos\theta\exp(\alpha\cos\theta) \tag{24}$$

$$f(\rho,\beta)=\frac{-K}{2\alpha}\rho\cos\beta\exp(\alpha\rho\cos\beta) \tag{25}$$

$$f(\rho,\gamma)=\frac{-K}{2\alpha}\rho\cos\gamma\exp(\alpha\rho\cos\gamma). \tag{26}$$

Now, we shall find the Green function for equation (5) under the following boundary conditions:

$$G(\boldsymbol{x};1,\theta')=0 \tag{27}$$

$$G(\boldsymbol{x};\rho',\beta)=0 \tag{28}$$

$$G(\boldsymbol{x};\rho',\gamma)=0. \tag{29}$$

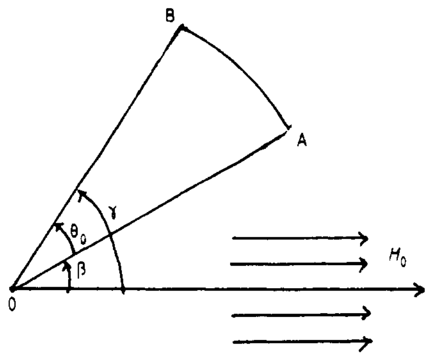


**Figure 1.**

For this purpose, we expand the required functions into a series only with sines by the following rules:

$$h(\theta') = \sum_{n=1}^{\infty} b_n \sin \nu_n(\theta' - \beta) \tag{30}$$

$$b_n = \frac{2}{\theta_0} \int_{\beta}^{\gamma} h(\theta') \sin \nu_n(\theta' - \beta)\, \mathrm{d}\theta' \tag{31}$$

where $\nu_n = (\pi/\theta_0)n$. Hence the $\theta$ delta function can be written as

$$\delta(\theta - \theta') = \frac{2}{\theta_0} \sum_{n=1}^{\infty} \sin \nu_n(\theta - \beta) \sin \nu_n(\theta' - \beta) \tag{32}$$

and the Green function $G(\boldsymbol{x}, \boldsymbol{x}')$ can be written as

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{2}{\theta_0} \sum_{n=1}^{\infty} \sin \nu_n(\theta - \beta) \sin \nu_n(\theta' - \beta) g_n(\rho, \rho'). \tag{33}$$

Substituting (32) and (33) in (5) we obtain an equation for $g_n(\rho, \rho')$:

$$\frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}g_n}{\mathrm{d}\rho}\right) - \left(\alpha^2 + \frac{\nu_n^2}{\rho^2}\right) g_n = -\frac{4\pi}{\rho} \delta(\rho - \rho'). \tag{34}$$

Solution of (34) can be written as

$$g_n(\rho, \rho') = A_n I_{\nu_n}(\alpha\rho_<)[K_{\nu_n}(\alpha\rho_>) + B_n I_{\nu_n}(\alpha\rho_>)] \tag{35}$$

where $B_n = -K_{\nu_n}(\alpha)/I_{\nu_n}(\alpha)$. By the same method given in the last section, we find the constant

$$A_n = 4\pi. \tag{36}$$

The required Green function is therefore

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{8\pi}{\theta_0} \sum_{n=1}^{\infty} \sin \nu_n(\theta - \beta) \sin \nu_n(\theta' - \beta) I_{\nu_n}(\alpha\rho_<)[K_{\nu_n}(\alpha\rho_>) + B_n I_{\nu_n}(\alpha\rho_>)]. \tag{37}$$

Now we proceed to calculate $\partial G/\partial n'$ on the boundary surface. On arc $\widehat{\mathrm{AB}}$, similar to the method of obtaining (16) from (13), from (37) we obtain

$$a\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial \rho'}\bigg|_{\rho'=1} = -\frac{8\pi}{\theta_0} \sum_{n=1}^{\infty} \sin \nu_n(\theta - \beta) \sin \nu_n(\theta' - \beta) \frac{I_{\nu_n}(\alpha\rho)}{I_{\nu_n}(\alpha)}. \tag{38}$$

On line OB, from (37) we obtain

$$\begin{aligned} a\frac{\partial G}{\partial n'} &= \frac{1}{\rho'} \frac{\partial G}{\partial \theta'}\bigg|_{\theta'=\gamma} \\ &= \frac{8\pi}{\rho'\theta_0} \sum_{n=1}^{\infty} (-1)^n \nu_n \sin \nu_n(\theta - \beta) I_{\nu_n}(\alpha\rho_<)[K_{\nu_n}(\alpha\rho_>) + B_n I_{\nu_n}(\alpha\rho_>)]. \end{aligned} \tag{39}$$

On line OA, $a(\partial G/\partial n')$ has the same form as on line OB except that $(-1)^n$ is replaced by unity, i.e.

$$\begin{aligned} a\frac{\partial G}{\partial n'} &= -\frac{1}{\rho'} \frac{\partial G}{\partial \theta'}\bigg|_{\theta'=\beta} \\ &= -\frac{8\pi}{\rho'\theta_0} \sum_{n=1}^{\infty} \nu_n \sin \nu_n(\theta - \beta) I_{\nu_n}(\alpha\rho_<)[K_{\nu_n}(\alpha\rho_>) + B_n I_{\nu_n}(\alpha\rho_>)]. \end{aligned} \tag{40}$$

Now we can calculate $f(\rho, \theta)$ from (17). In this case the boundary surface $S$ is composed of three parts, i.e. arc $\widehat{AB}$, line OA, line OB, and the surface integral in (17) is composed of three integrals, represented by D(AB), D(OA), D(OB), respectively. Substituting (24) and (38) in (17), we obtain

$$\mathrm{D(AB)} = \frac{4K}{\alpha} \sum_{n=1}^{\infty} \frac{W_n(\alpha, \beta)}{I_{\nu_n}(\alpha)} I_{\nu_n}(\alpha\rho) \sin \nu_n(\theta - \beta) \tag{41}$$

where $W_n(\alpha, \beta)$ represents the following integral:

$$W_n(\alpha, \beta) = \int_0^{\pi} \sin n\phi \cos\left(\frac{n}{\nu_n}\phi + \beta\right) \exp\left[\alpha \cos\left(\frac{n}{\nu_n}\phi + \beta\right)\right] \mathrm{d}\phi. \tag{42}$$

Substituting (25) and (40) in (17), we obtain

$$\mathrm{D(OA)} = \left(\frac{K}{2\alpha}\cos\beta\right) \frac{8\pi}{\theta_0} \sum_{n=1}^{\infty} \nu_n \sin \nu_n(\theta - \beta)$$
$$\times [B_n I_{\nu_n}(\alpha\rho) N_n(\alpha, \beta) + K_{\nu_n}(\alpha\rho) U_n^{\alpha}(\rho, \beta) + I_{\nu_n}(\alpha\rho) V_n^{\alpha}(\rho, \beta)] \tag{43}$$

where $N_n(\alpha, \psi)$, $U_n^{\alpha}(\rho, \psi)$, $V_n^{\alpha}(\rho, \psi)$ represent the following integrals, respectively:

$$N_n(\alpha, \psi) = \int_0^1 \exp(\alpha\rho' \cos\psi) I_{\nu_n}(\alpha\rho')\, \mathrm{d}\rho' \tag{44}$$

$$U_n^{\alpha}(\rho, \psi) = \int_0^{\rho} \exp(\alpha\rho' \cos\psi) I_{\nu_n}(\alpha\rho')\, \mathrm{d}\rho' \tag{45}$$

$$V_n^{\alpha}(\rho, \psi) = \int_{\rho}^1 \exp(\alpha\rho' \cos\psi) K_{\nu_n}(\alpha\rho')\, \mathrm{d}\rho'. \tag{46}$$

Substitute (26) and (39) in (17), we obtain

$$\mathrm{D(OB)} = \left(\frac{-K}{2\alpha}\cos\gamma\right) \sum_{n=1}^{\infty} (-1)^n \frac{8\pi}{\theta_0} \nu_n \sin \nu_n(\theta - \beta)$$
$$\times [B_n I_{\nu_n}(\alpha\rho) N_n(\alpha, \gamma) + K_{\nu_n}(\alpha\rho) U_n^{\alpha}(\rho, \gamma) + I_{\nu_n}(\alpha\rho) V_n^{\alpha}(\rho, \gamma)]. \tag{47}$$

Hence, we obtain

$$f(\rho, \theta) = -\frac{1}{4\pi}[\mathrm{D(AB)} + \mathrm{D(OA)} + \mathrm{D(OB)}]$$
$$= \frac{-K}{2\alpha}\left(\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{W_n(\alpha, \beta)}{I_{\nu_n}(\alpha)} I_{\nu_n}(\alpha\rho) \sin \nu_n(\theta - \beta) + \frac{2}{\theta_0}\cos\beta \sum_{n=1}^{\infty} \nu_n \sin \nu_n(\theta - \beta)\right.$$
$$\times [B_n I_{\nu_n}(\alpha\rho) N_n(\alpha, \beta) + K_{\nu_n}(\alpha\rho) U_n^{\alpha}(\rho, \beta) + I_{\nu_n}(\alpha\rho) V_n^{\alpha}(\rho, \beta)]$$
$$- \frac{2}{\theta_0}\cos\gamma \sum_{n=1}^{\infty} (-1)^n \nu_n \sin \nu_n(\theta - \beta)$$
$$\left.\times [B_n I_{\nu_n}(\alpha\rho) N_n(\alpha, \gamma) + K_{\nu_n}(\alpha\rho) U_n^{\alpha}(\rho, \gamma) + I_{\nu_n}(\alpha\rho) V_n^{\alpha}(\rho, \gamma)]\right). \tag{48}$$

$g(\rho, \theta)$ has the same form as $f(\rho, \theta)$, except that there is no negative sign in the front of the equation and $W_n$, $N_n$, $U_n$, $V_n$ are replaced by $\bar{W}_n$, $\bar{N}_n$, $\bar{U}_n$, $\bar{V}_n$ where the last four symbols differ from the first four only in adding a negative sign in the exponential arguments, since the boundary condition for $g$ differs from that for $f$ only in the sign of $\alpha$ and $\partial G/\partial n'$ is independent of the sign of $\alpha$ (as in (37) $\alpha$ may be replaced by $|\alpha|$ since only $\alpha^2$ appears in (34)).

## 4. Conclusions

The application of the Green function methods to MHD pipe flow enable explicit analytic solutions to be obtained for any sector cross section. These solutions for special geometries are also particularly useful in checking out computer codes intended for more complex geometries.

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